

ESGtoolkit, tools for Economic Scenarios Generation

Thierry Moudiki
13th June 2014

Contents

1 Overview	1
1.1 Context	1
1.2 <code>simdiff</code>	2
1.3 <code>simshocks</code>	5
2 Examples	6
2.1 Generating dependent shocks with <code>simshocks</code>	6
2.2 Example with <code>simdiff</code> and <code>simshocks</code> : Option pricing under the Bates model (SVJD) for equity	9

1 Overview

1.1 Context

An *Economic Scenario Generator* (ESG) is a tool for projection of plausible future paths for an insurer's financial risk factors. It helps her in pricing her insurance products, and in assessing her current and future solvency.

Two types of ESGs are generally needed, for different purposes : a *real-world* ESG, and a *market consistent* ESG.

The aim of a real-world ESG is to produce projections of risk factors, whose distribution patterns are coherent with the past distribution of those risk factors. Real-world scenarios are mainly used for the valuation of solvency capital requirements.

A market consistent ESG shall produce projections of risk factors that are coherent with market prices observed at the valuation date. Market consistent scenarios are mainly used for the best estimate valuation of the technical reserves.

Hence, in real-world simulations the historical probability is used and in market consistent simulations, the projection of risk factors is made in a risk-neutral probability. A risk-neutral probability measure is a measure under which the discounted prices of assets are martingales.

A simple example about transitioning from a simulation under the historical probability to a simulation under a risk-neutral probability can be made

by using the Black-Scholes model, a geometric Brownian motion. In real-world simulations, the asset evolves according to the following SDE¹ (with a drift μ , a volatility σ , and $(W(t))_{t \geq 0}$ being a standard brownian motion) :

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (1)$$

Let r be a constant risk-free rate. $e^{-rt}S(t)$, the discounted price of $S(t)$, will be a martingale if

$$d(e^{-rt}S(t)) \quad (2)$$

is driftless.

Applying Ito's formula to $e^{-rt}S(t)$, we have :

$$d(e^{-rt}S(t)) = -re^{-rt}S(t)dt + e^{-rt}dS(t) - \frac{1}{2} \cdot 0 < dS(t), dS(t) > \quad (3)$$

$$= -re^{-rt}S(t)dt + e^{-rt}\mu S(t)dt + e^{-rt}S(t)\sigma dW(t) \quad (4)$$

$$= e^{-rt}S(t) [(\mu - r)dt + \sigma dW(t)] \quad (5)$$

Thus, the drift vanishes iff $\mu = r$, that is, if the asset with price $S(t)$ rewards the risk-free rate r . Under this martingale probability measure, the asset price can thus be re-written as :

$$dS(t) = rS(t)dt + \sigma S(t)dW^*(t) \quad (6)$$

Where $(W^*(t))_{t \geq 0}$ is a standard brownian motion under the risk-neutral measure.

ESGtoolkit does not directly provide multiple asset models, but instead, some building blocks for constructing a variety of these.

Two main functions are therefore provided : `simshocks`, `simdiff`.

In this vignette, I introduce these functions, and present how they could be used in building other models. Others tools for statistical testing and visualization are introduced as well.

As a reminder : there are no perfect models, and the more sophisticated doesn't necessarily mean the most judicious. To avoid possible disasters, it's important to know precisely the strengths and weaknesses of a model before using it.

1.2 `simdiff`

Let $(W(t))_{t \geq 0}$ be a standard brownian motion. `simdiff` makes simulations of a diffusion process $(X(t))_{t \geq 0}$, which evolves according to the following equation :

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) + \gamma(t, X(t-), J)dN(t) \quad (7)$$

¹Stochastic Differential Equation

Actually, this is just a generic formulation of the models from `simdiff`. **Not all the parts of this expression are required all the times**, but only $\sigma(t, X(t))dW(t)$.

The part $\gamma(t, X(t-), J)dN(t)$ in particular, is optional, and not available for all the models. It contains jumps of the process, that occur according to a homogeneous Poisson process $(N(t))_{t \geq 0}$ with intensity λ . The time elapsed between two jumping times follows an exponential $\epsilon(\lambda)$ distribution; and the number of jumps of the process on $[0, t[$ follows a Poisson distribution $\mathcal{P}(\lambda t)$. The magnitude of the jumps is controlled by J .

Let's make this clearer now. The basic building blocks models implemented in `simdiff`, are :

- An Ornsstein-Uhlenbeck process; for `simdiff` used with parameter `model = "OU"`, and parameters `theta1`, `theta2` and `theta3` provided (if `theta1` or `theta2` are not necessary for building the model, they are to be provided and set to 0) :

$$\begin{aligned}\mu(t, X(t)) &= (\theta_1 - \theta_2 X(t)) \\ \sigma(t, X(t)) &= \theta_3\end{aligned}$$

- A Cox-Ingersoll-Ross process; for `simdiff` used with parameter `model = "CIR"`, and parameters `theta1`, `theta2`, `theta3` provided (if `theta1` or `theta2` are not necessary for building the model, they are to be provided and set to 0) :

$$\begin{aligned}\mu(t, X(t)) &= (\theta_1 - \theta_2 X(t)) \\ \sigma(t, X(t)) &= \theta_3 \sqrt{X(t)}\end{aligned}$$

- A Geometric Brownian motion, or 'augmented' versions; for `simdiff` used with parameter `model = "GBM"`, and parameters `theta1`, `theta2`, `theta3` provided. For the sake of clarity, the argument `model` is set to "GBM", but not only the Geometric Brownian motion with constant parameters is available. We can have :

A Geometric Brownian Motion

$$\begin{aligned}\mu(t, X(t)) &= \theta_1 X(t) \\ \sigma(t, X(t)) &= \theta_2 X(t)\end{aligned}$$

A modified Geometric Brownian Motion, with time-varying drift and constant volatility

$$\begin{aligned}\mu(t, X(t)) &= \theta_1(t)X(t) \\ \sigma(t, X(t)) &= \theta_2 X(t)\end{aligned}$$

A modified Geometric Brownian Motion, with time-varying volatility and constant drift

$$\begin{aligned}\mu(t, X(t)) &= \theta_1 X(t) \\ \sigma(t, X(t)) &= \theta_2(t)X(t)\end{aligned}$$

It's technically possible to have both θ_1 and θ_2 varying with time (both provided as multivariate time series). But it's not advisable to do this, unless you know exactly why you're doing it.

Jumps are available only for `model = "GBM"`. The jumps arising from the Poisson process have a common magnitude $J = 1 + Z$, whose distribution ν is either lognormal or double-exponential. Between two jumps, the process behaves like a Geometric Brownian motion, and at jumping times, it increases by $Z\%$.

For lognormal jumps (Merton model), the distribution ν of J is :

$$\log(J) = \log(1 + Z) \sim \mathcal{N}(\log(1 + \mu_Z) - \frac{\sigma_Z^2}{2}, \sigma_Z^2) \quad (8)$$

For double exponential jumps (Kou's model), the distribution ν of J is :

$$\log(J) = \log(1 + Z) \sim \nu(dy) = p \frac{1}{\eta_u} e^{-\frac{1}{\eta_u} y} \mathbb{1}_{y>0} + (1 - p) \frac{1}{\eta_d} e^{\frac{1}{\eta_d} y} \mathbb{1}_{y<0} \quad (9)$$

Hence for taking jumps into account when `model = "GBM"`, optional parameters are to be provided to `simdiff`, namely :

- `lambda` the intensity of the Poisson process
- `mu.z` the average jump magnitude (only for **lognormal** jumps)
- `sigma.z` the standard deviation of the jump magnitude (only for **lognormal** jumps)
- `p` the probability of positive jumps (only for **double exponential** jumps)
- `eta_up` the mean of positive jumps (only for **double exponential** jumps)
- `eta_down` the mean of negative jumps (only for **double exponential** jumps)

`simdiff`'s core loops are written in C++ *via* `Rcpp`, for an enhanced performance. Currently, for the simulation of the Ornstein-Uhlenbeck process, with `model = "OU"`, the simulation of a Cox-Ingersoll-Ross process with `model = "CIR"`, or a geometric brownian motion with `model = "GBM"`, it uses an exact simulation (please see the references for details), which means there's no discretization of the processes.

In `simdiff`, the user can choose the horizon of the projection and the sampling frequency (annual, semi-annual, quarterly ...). The output is a time series object created by `ts()` from base R.

For a customized simulation of the $\epsilon \sim \mathcal{N}(0,1)$ embedded in the SDE expression *via* $dW(t) = \epsilon dt$, one can fill `simdiff`'s parameter `eps`, with an output of the function `simshocks`. `simshocks` is described in the next section.

1.3 `simshocks`

`simshocks` is the complementary function to `simdiff`, with which you can simulate custom the $\epsilon \sim \mathcal{N}(0,1)$ (that we call shocks) embedded into the diffusion as :

$$dW(t) = \epsilon dt \tag{10}$$

For the simulation of gaussian increments of a univariate process `simshocks` is written in C++ *via* `Rcpp`.

When it comes to the simulation of multi-factors models, or the simulation of risk factors with flexible dependence structure, `simshocks` calls the underlying function `CDVinesim`, from the package `CDVine`. `CDVineSim` makes simulations of canonical (**C-vine**) and **D-vine copulas**.

Simply put, a copula is a function which gives a multidimensional distribution to given margins. If $(X_1, \dots, X_d)^T$ is a random vector with margins of cumulative distribution functions F_1, \dots, F_d , there exists a copula function C , such that the d-dimensional cumulative distribution function of $(X_1, \dots, X_d)^T$ is :

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \tag{11}$$

If the marginal distributions F_1, \dots, F_d are continuous, then C is unique.

On the other hand, if C is a copula, and F_1, \dots, F_d are 1-dimensional cumulative distribution functions, the previous equation defines a joint cumulative distribution function for $(X_1, \dots, X_d)^T$, with margins F_1, \dots, F_d .

Contrarily to the multivariate Gaussian or Student-t copulas, vine copulas accurately model the dependence in high dimensions. They use the density functions of bivariate copulas (called pair-copula) to iteratively build a multivariate density function, which leads to a great flexibility in modeling the dependence.

`simshocks` applies inverse standard gaussian cumulative distribution function to the uniform margins of `CDVinesim` to obtains gaussian shocks, with various dependence structures between them.

The package `CDVineSim` can be used first, to choose the copula, and make an inference on it. Sometimes, the choice of the relevant copula is also made with expert knowledge.

2 Examples

2.1 Generating dependent shocks with `simshocks`

To use `simshocks`, you need to specify the number of simulations `n` that you need, the type of dependence family, and additional parameters, depending on the copula that you use. For a simulation of Gaussian the copula, the family is 1 :

```
library(ESGtoolkit)

# Number of simulations
nb <- 1000

# Number of risk factors
d <- 2

# Number of possible combinations of the risk factors (here : 1)
dd <- d * (d - 1)/2

# Family : Gaussian copula
fam1 <- rep(1, dd)
# Correlation coefficients between the risk factors (d*(d-1)/2)
par0.1 <- 0.1
par0.2 <- -0.9
```

The correlation coefficient is provided through the argument `par` :

```
set.seed(2)
# Simulation of shocks for the d risk factors
s0.par1 <- simshocks(n = nb, horizon = 4, family = fam1, par = par0.1)

s0.par2 <- simshocks(n = nb, horizon = 4, family = fam1, par = par0.2)
```

You can make a correlation test with `esgcortest`, to assess whether the correlation estimate is significantly close to the correlation that you specified, or not.

If the confidence interval contains the true value at a given confidence level, then the null hypothesis chosen is not to be rejected at this level.

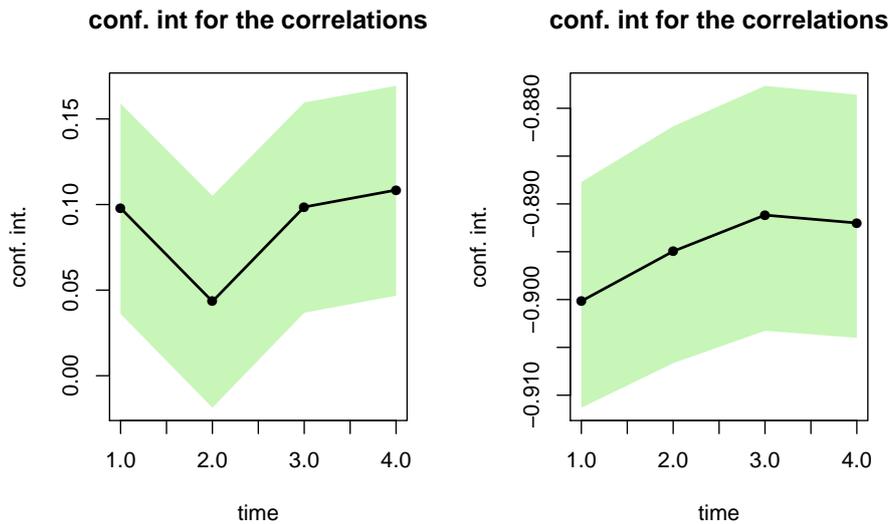
Use `simshocks` along with `set.seed`, so that when the correlation seems acceptable, you can reproduce the simulation.

```
# Correlation test
esgcor.test(s0.par1)

## $cor.estimate
## Time Series:
## Start = 1
## End = 4
## Frequency = 1
## [1] 0.09793 0.04339 0.09855 0.10845
##
## $conf.int
## Time Series:
## Start = 1
## End = 4
## Frequency = 1
## Series 1 Series 2
## 1 0.03616 0.1590
## 2 -0.01865 0.1051
## 3 0.03679 0.1596
## 4 0.04677 0.1693
```

These confidence intervals on the estimated correlations can also be visualized with `esgplotbands`:

```
test <- esgcor.test(s0.par2)
par(mfrow = c(1, 2))
esgplotbands(esgcor.test(s0.par1))
esgplotbands(test)
```



Now with other types of dependences, namely rotated versions of the Clayton copula :

```
# Family : Rotated Clayton (180 degrees)
fam2 <- 13
par0.3 <- 2

# Family : Rotated Clayton (90 degrees)
fam3 <- 23
par0.4 <- -2

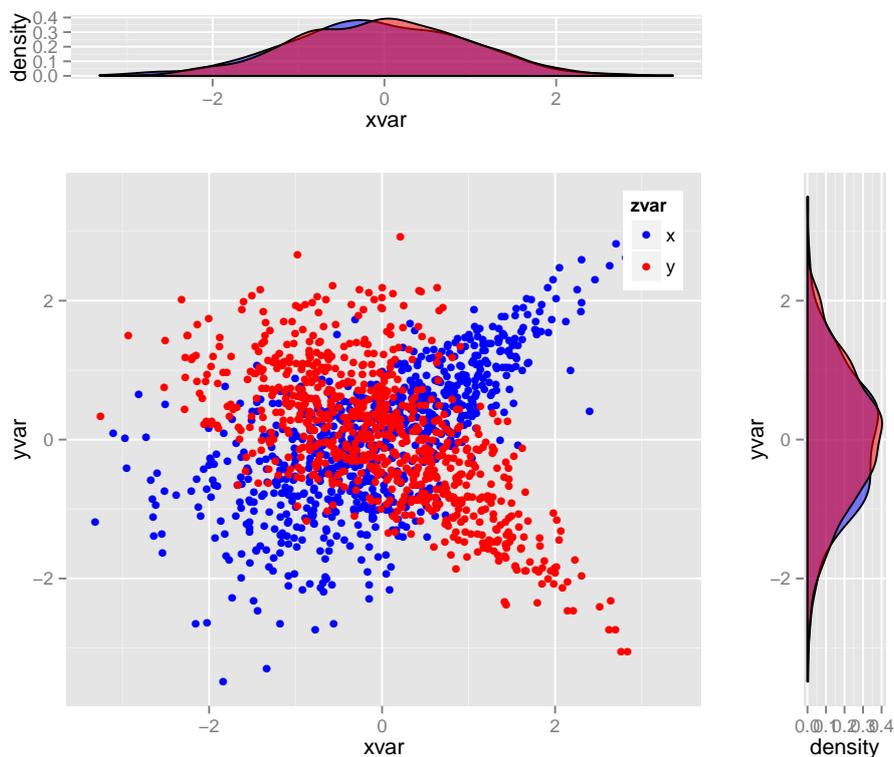
# number of simulations
nb <- 200

# Simulation of shocks for the d risk factors
s0.par3 <- simshocks(n = nb, horizon = 4, family = fam2, par = par0.3)

s0.par4 <- simshocks(n = nb, horizon = 4, family = fam3, par = par0.4)
```

There's a nice function from the package, `esgplotshocks`, that helps you in visualizing the dependence between the shocks (inspired by [this blog post](#)):

```
esgplotshocks(s0.par3, s0.par4)
```



2.2 Example with `simdiff` and `simshocks` : Option pricing under the Bates model (SVJD) for equity

SVJD stands for Stochastic Volatility with Jump Diffusion. In this model, the volatility of the asset's price evolves as a CIR process. The price itself is a Geometric Brownian motion between jumps, arising from a Poisson process. Here, we consider jumps with lognormal magnitude.

The model

$$\begin{aligned} dS(t) &= (r - \lambda\mu_Z)S(t)dt + \sqrt{v(t)}S(t)dW(t)^{(1)} + (J - 1)dN(t) \\ dv(t) &= \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW(t)^{(2)} \\ dW(t)^{(1)}dW(t)^{(2)} &= \rho dt \end{aligned}$$

We use the package `fOptions` to compute options' prices from market implied volatility :

```
library(fOptions)
```

The parameters of the Bates model are :

```
# Spot variance
V0 <- 0.1372
# mean-reversion speed
kappa <- 9.511/100
# long-term variance
theta <- 0.0285
# volatility of volatility
volvol <- 0.801/100
# Correlation between stoch. vol and prices
rho <- -0.5483
# Intensity of the Poisson process
lambda <- 0.3635
# mean and vol of the merton jumps diffusion
mu.J <- -0.2459
sigma.J <- 0.2547/100
m <- exp(mu.J + 0.5 * (sigma.J^2)) - 1
# Initial stock price
S0 <- 4468.17
# Initial short rate
r0 <- 0.0357
```

Now we make 300 simulations of shocks and diffusions, on a weekly basis, from today, up to year 1. The shocks are simulated by using a variance reduction technique : antithetic variates (argument `method`).

```
n <- 300
horizon <- 1
```

```

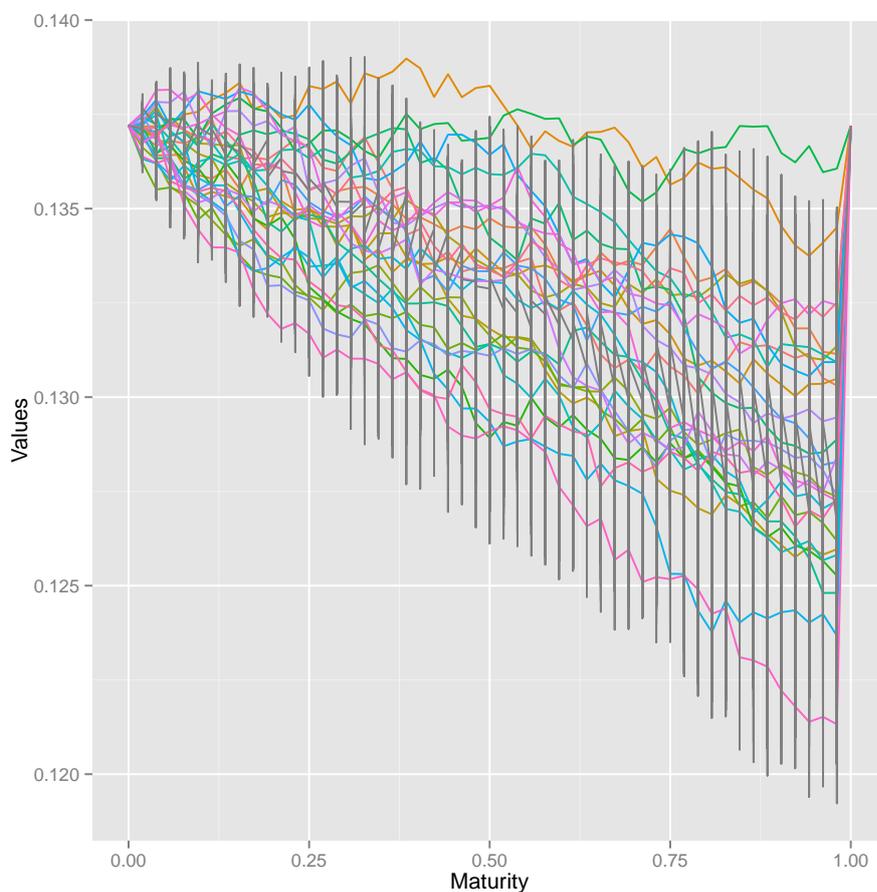
freq <- "weekly"

# Simulation of shocks, with antithetic variates
shocks <- simshocks(n = n, horizon = horizon, frequency = freq, method = "anti",
  family = 1, par = rho)

# Vol simulation
sim.vol <- simdiff(n = n, horizon = horizon, frequency = freq, model = "CIR",
  x0 = V0, theta1 = kappa * theta, theta2 = kappa, theta3 = volvol, eps = shocks[[1]])

# Plotting the volatility (only for a low number of simulations)
esgplotts(sim.vol)

```



Finally, the price's simulation takes **exactly the same parameters** n , $horizon$, $frequency$ as `simshocks` and `simdiff`, and the volatility is embedded through θ_2 .

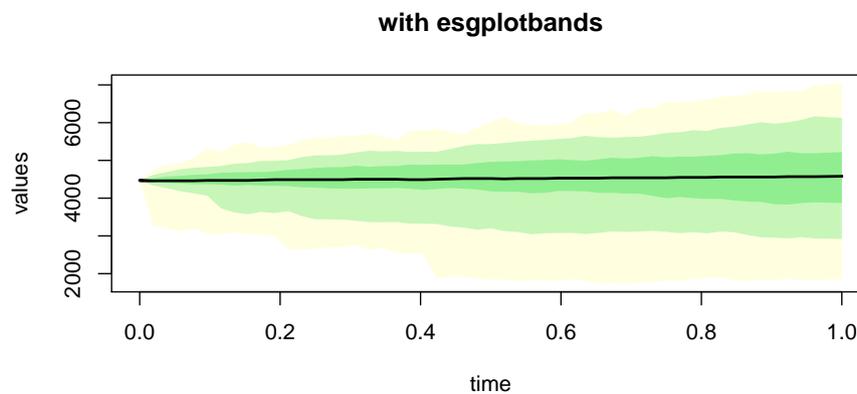
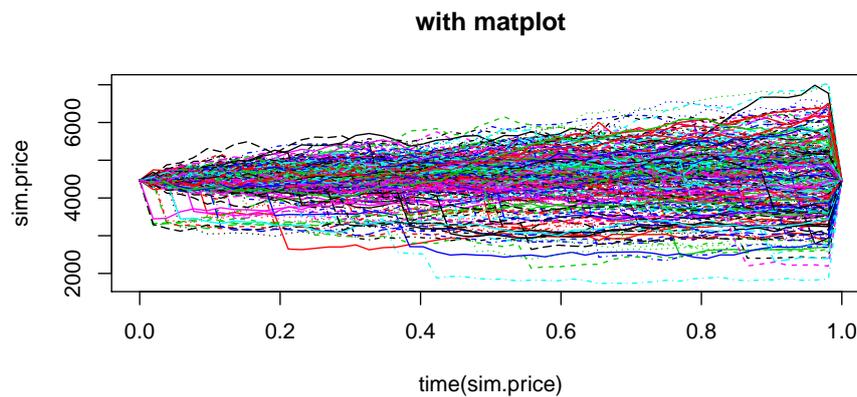
```

# prices simulation
sim.price <- simdiff(n = n, horizon = horizon, frequency = freq, model = "GBM",
  x0 = S0, theta1 = r0 - lambda * m, theta2 = sim.vol, lambda = lambda, mu.z = mu.J,
  sigma.z = sigma.J, eps = shocks[[2]])

```

We can clearly see the prices jumping with `matplot`. But `esgplotbands`, offering a view of the paths by percentiles, will be more useful for thousands of simulations :

```
par(mfrow = c(2, 1))
matplot(time(sim.price), sim.price, type = "l", main = "with matplot")
esgplotbands(sim.price, main = "with esgplotbands", xlab = "time", ylab = "values")
```



Now, we would like to verify the convergence of the estimated discounted prices to the initial asset price :

$$\frac{1}{N} \sum_{i=1}^N e^{-rT} S_T^{(i)} \longrightarrow \mathbb{E}[e^{-rT} S_T] = S_0 \quad (12)$$

where N is the number of simulations, r is the constant risk free rate, and T is a maturity of 2 weeks.

```
# Discounted Monte Carlo price
as.numeric(esgmcprices(r0, sim.price, 2/52))

## [1] 4456
```

```
# Initial price
S0

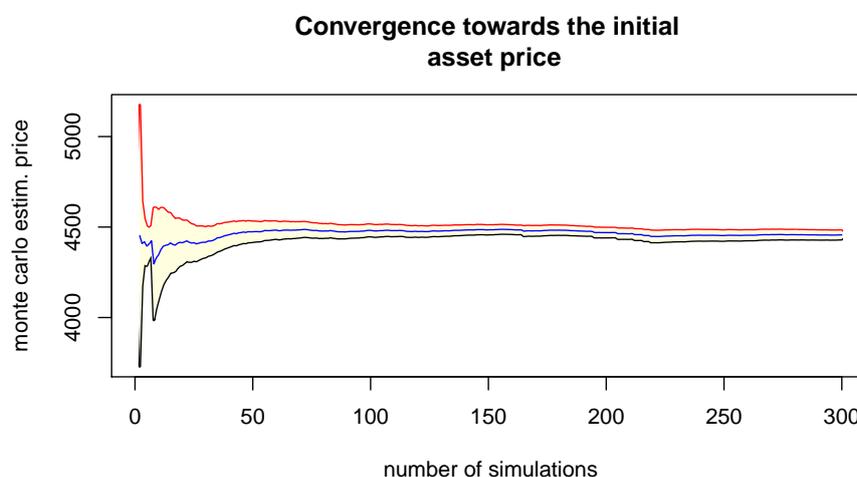
## [1] 4468

# pct. difference
as.numeric((esgmcprices(r0, sim.price, 2/52)/S0 - 1) * 100)

## [1] -0.2612
```

One would also want to see how fast is the convergence towards S_0 :

```
# convergence of the discounted price
esgmccv(r0, sim.price, 2/52, main = "Convergence towards the initial \n asset price")
```



`esgmcprices` and `esgmccv` give information about the mean, but a statistical test gives more information.

```
martingaletest.sim.price <- esgmartingaletest(r = r0, X = sim.price, p0 = S0)
```

`esgmartingaletest` computes for each T , a Student's t-test of

$$H_0 : \mathbb{E}[e^{-rT} S_T - S_0] = 0$$

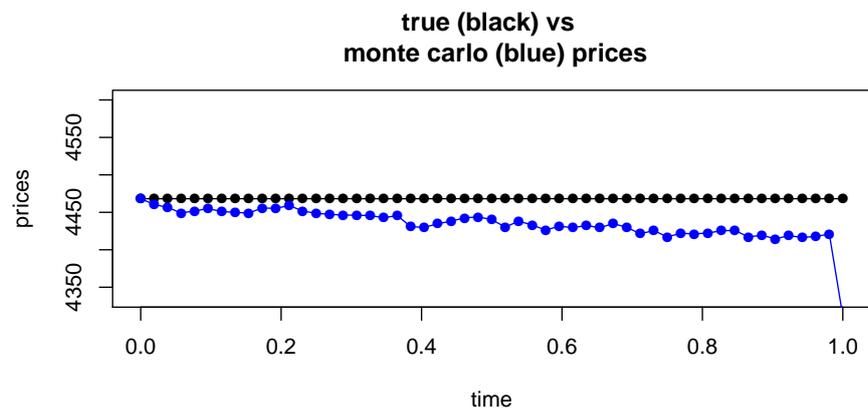
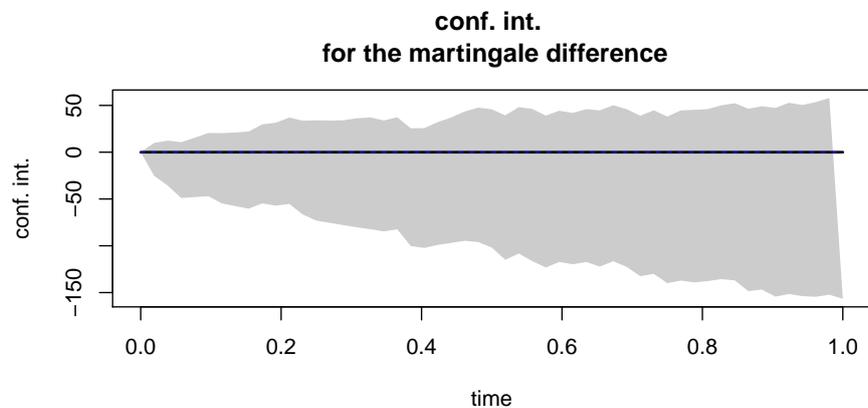
versus the alternative hypothesis that the mean is not 0, at a given confidence level (default is 95%).

`esgmartingaletest` also provides p-values, and confidence intervals for the mean value.

If all the confidence intervals contain 0, then the null hypothesis is not rejected at the given level, let's say 95%. Which means that there are less than 5 chances out of 100 to be wrong by saying that the true mean of the distribution is 0.

`esgplotbands` gives a visualization of the confidence intervals, as well as the average discounted prices.

```
esgplotbands(martingaletest.sim.price)
```



Now, we price a call option under the Bates model :

```
# Option pricing
# Strike
K <- 3400
Kts <- ts(matrix(K, nrow(sim.price), ncol(sim.price)), start = start(sim.price),
           deltat = deltat(sim.price), end = end(sim.price))

# Implied volatility
sigma.imp <- 0.6625

# Maturity
maturity <- 2/52

# payoff at maturity
payoff <- (sim.price - Kts) * (sim.price > Kts)
payoff <- window(payoff, start = deltat(sim.price), deltat = deltat(sim.price),
                 names = paste0("Series ", 1:n))
```

```

# True price
c0 <- GBSOption("c", S = S0, X = K, Time = maturity, r = r0, b = 0, sigma = sigma.imp)
c0@price

## [1] 1070

# Monte Carlo price
as.numeric(esgmcprices(r = r0, X = payoff, maturity))

## [1] 1063

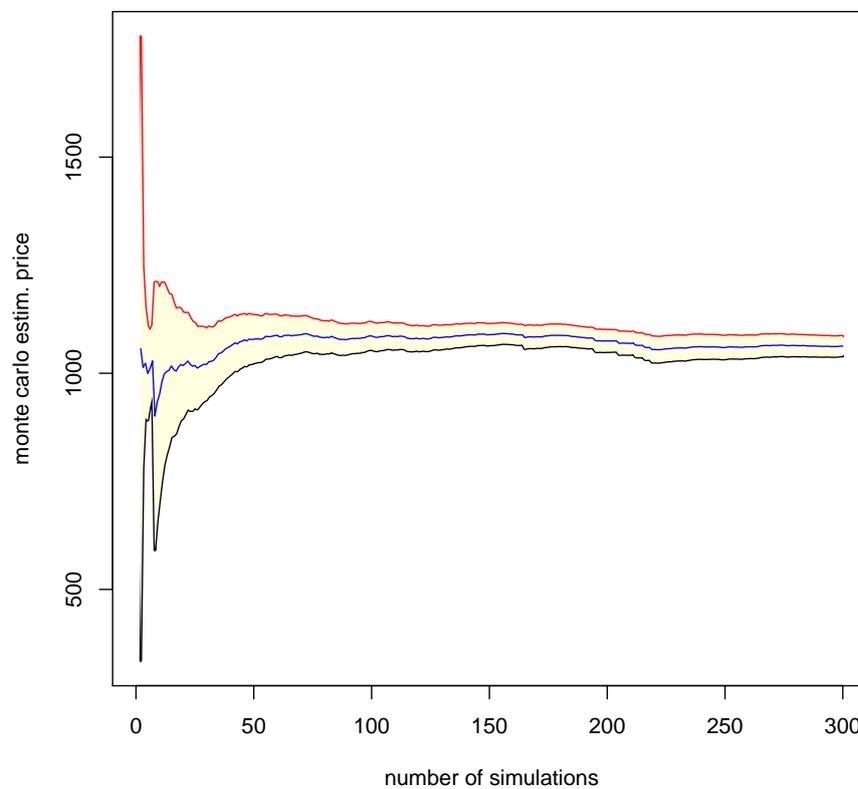
# pct. difference
as.numeric((esgmcprices(r = r0, X = payoff, maturity = maturity)/c0@price -
1) * 100)

## [1] -0.6772

# Convergence towards the option price
esgmcv(r = r0, X = payoff, maturity = maturity, main = "Convergence towards the call \n o

```

**Convergence towards the call
option price**



References

- Bates DS (1996). "Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options." *Review of financial studies*, 9(1), 69–107.
- Black F, Scholes M (1973). "The pricing of options and corporate liabilities." *The journal of political economy*, pp. 637–654.
- Brechmann EC, Czado C (2012). "Risk management with high-dimensional vine copulas: An analysis of the Euro Stoxx 50."
- Brechmann EC, Schepsmeier U (2013). "Modeling dependence with C-and D-vine copulas: The R-package CDVine." *Journal of Statistical Software*, 5(3), 1–27.
- Brigo D, Mercurio F (2006). *Interest rate models-theory and practice: with smile, inflation and credit*. Springer.
- Cox JC, Ingersoll Jr JE, Ross SA (1985). "A theory of the term structure of interest rates." *Econometrica: Journal of the Econometric Society*, pp. 385–407.
- Eddelbuettel D, François R (2011). "Rcpp: Seamless R and C++ integration." *Journal of Statistical Software*, 40(8), 1–18.
- Glasserman P (2004). *Monte Carlo methods in financial engineering*, volume 53. Springer.
- Iacus SM (2008). *Simulation and inference for stochastic differential equations: with R examples*. Springer.
- Kou SG (2002). "A jump-diffusion model for option pricing." *Management science*, 48(8), 1086–1101.
- Merton RC (1976). "Option pricing when underlying stock returns are discontinuous." *Journal of financial economics*, 3(1), 125–144.
- Uhlenbeck GE, Ornstein LS (1930). "On the theory of the Brownian motion." *Physical review*, 36(5), 823.
- Vasicek O (1977). "An equilibrium characterization of the term structure." *Journal of financial economics*, 5(2), 177–188.
- Wickham H (2009). *ggplot2: elegant graphics for data analysis*. Springer.